

# Streamer formation and collapse in electron temperature gradient driven turbulence

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A simple model is useful to understand the formation and persistence of radially elongated structures (streamers) in electron temperature gradient (ETG) driven modes. The ETG model is very similar to the thermal Rossby wave model, a system of broad interest. The detailed correspondence of these two models is discussed. Streamer formation in this simple model is analyzed using the modulational stability method. In the inviscid limit of the model, an amplitude equation similar to the nonlinear Schrödinger equation (NLS) is derived. This equation has a second derivative cubic nonlinearity and is identified as a special case of a more general higher order NLS. Analytical solutions are found in the form of travelling waves and a localized thorn. Using the Lagrangian structure of the amplitude equation, it is shown that one-dimensional collapse in the poloidal direction is possible in this system for certain parameter values, and for sufficiently localized initial flow. This identifies a parameter regime basin in which there is an attractor with the structure of a thin extended streamer. In the viscous limit, another amplitude equation, which is a certain special case of the generalized complex Ginzburg–Landau equation, is obtained. Fixed points of the corresponding dynamical system are identified and their stability is investigated. © 2004 American Institute of Physics. [DOI: 10.1063/1.1637920]

## I. INTRODUCTION

### A. Motivation

Understanding the formation of large scale structures is a major challenge for predicting particle and heat transport in tokamak devices. Although computers can simulate these structures by computing  $n$ -particle interactions from first principles, we still need insight gained from basic explanations and simplified physical pictures to understand the phenomena that are observed in simulations. In this sense, a secondary instability analysis may prove useful in gaining insight in to the nonlinear behavior of the electron temperature gradient driven (ETG) modes.

The ETG mode<sup>1,2</sup> is one of the possible explanations of the observed high level electron transport in tokamaks. It is a quasifluid electron mode, with dynamics akin to the ion temperature gradient driven mode (ITG) (Ref. 3) ( $k_{\perp} \rho_i \gg 1, \tilde{n} = n - e\tilde{\Phi}/T_i$ ). Streamers, which are large scale structures ( $q_y \ll k_y$ ) that arise from dynamical instabilities driven by underlying small scale turbulence, have been observed in some of the computer simulations<sup>4–6</sup> of both modes. They correspond to the radially elongated limit ( $q_y \gg q_x$ ), of a more general class of structures called convective cells.<sup>7</sup> Thus, streamers are the opposite limit of zonal flows which are elongated in poloidal direction (i.e.,  $q_x \gg q_y$ ). However, similar structures may arise, also as a result of the anisotropy of the linear instability.

In this paper, we explore modulational instability of streamers, in ETG turbulence using the thermal Rossby wave (TR) model. Thermal Rossby waves<sup>8</sup> are Rossby waves

(similar to drift waves, see for example Ref. 9), driven unstable by buoyancy forces (similar to magnetic drifts), and stabilized by the  $\beta$ -effect (analogous to the diamagnetic frequency and diamagnetic stabilization). Thus, the structure of the TR instabilities is quite similar, indeed, to that of toroidal ETG/ITG modes. Therefore, they are a natural, “simplified” model for study of nonlinear ETG/ITG physics. A fundamental difference is the structure of dissipation. Viscosity plays an important role in Thermal Rossby waves, but ETG modes are primarily inviscid (see Table I for the correspondence between ETG and TR waves).

Large scale structures in these systems form as a result of coherent interaction of nonlinearly driven large scale flows and small scale ETG/TR fluctuations. Usually, it is local interaction (in  $k$  space) of small scale modes, which have a natural cut-off scale of order  $\rho_e \sqrt{T_i/T_e}$ , that are responsible for the mode coupling in cascade models. This is similar to avalanche formation through coupling of neighboring sites in models of cellular automata.<sup>10</sup>

In this paper, we limit ourselves to discussion of streamers.<sup>11</sup> The streamer paradigm is a possible explanation of the notion of extended cells and avalanches observed in gyrokinetic simulations.<sup>12</sup> It helps explain the excess of electron thermal transport, via bursty large scale events. Although streamers may also be a manifestation of the linear instability structure, their persistence in nonlinear regime motivates nonlinear stability calculations. Extended structures observed in geophysical systems, in the form of stratospheric intrusions into the troposphere,<sup>13</sup> may also have a similar origin.

The approach employed in this work does not explain how the turbulent structures become quasi-one-dimensional

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TABLE I. A summary of analogy between thermal rossby waves and drift-ETG modes.

Thermal Rossby	Drift-ETG
Geostrophic balance:	Electron drifts:
$v_0 = -\frac{1}{2}E\nabla \times \hat{z}\pi(x,y)$	$v = \frac{\hat{z} \times \nabla \Phi}{B} - \frac{\hat{z} \times \nabla P_e}{neB}$
Effect of temperature gradient:	Background diamagnetic drift:
$\frac{dT_0}{dx}$	$\epsilon_{*e} = \rho_e \sqrt{\tau_e} \frac{1}{P_{i0}} \frac{dP_{e0}}{dx} \hat{y}$
Variation of the rotation parameter: (Annulus analogon of the $\beta$ effect)	Diamagnetic stabilization:
$\frac{1}{\Omega} \frac{d\Omega}{dx}$	$\epsilon_{*i} = \rho_e \sqrt{\tau_e} \frac{1}{n_0} \frac{dn_0}{dx} \hat{y}$
Centrifugal buoyancy:	$\nabla B$ Drift:
$\Omega^2 r_0 = \frac{1}{\rho} \frac{dP_0}{dx}$	$v_B = \rho_e \sqrt{\tau_e} \left[ \hat{y} \cdot \left( \hat{x} \frac{1}{B} \frac{d}{dx} \right) \times (B \hat{z}) \right] \hat{y}$
Scaling of thermal Rossby	(Nondissipative) scaling of ETG
$x \rightarrow \frac{x}{D}$	$x \rightarrow \frac{x}{\rho_e \sqrt{\tau_e}}$
$t \rightarrow \frac{t\nu}{D^2}$	$t \rightarrow \epsilon_{*i} \Omega_e t$
$v \rightarrow \frac{Dv}{\nu}$	$\Phi \rightarrow \frac{1}{\epsilon_{*i}} \frac{e\Phi}{T_{i0}}$
$\Theta \rightarrow \frac{(T_2 - T_1)}{\mathcal{P}}$	$P \rightarrow \frac{\epsilon_B P_e}{\epsilon_{*i}^2 P_{i0}}$
	with dissipative ETG scaling
$\mathcal{R} = \gamma \frac{(T_2 - T_1) \Omega^2 r_0 D^3}{\nu \kappa}$	$\mathcal{R} = \frac{\epsilon_{*e} \epsilon_B}{\chi \nu} = \frac{\Omega_e^3 \rho_e^6 \tau_e^3}{\nu' \chi' L_p L_B}$
$\mathcal{P} = \frac{\nu}{\kappa}$	$\mathcal{P} = \frac{\nu}{\chi}$
$\eta_* = \frac{4D\eta_0}{LE}$	$S = \frac{\epsilon_{*i}}{\nu} = \frac{\Omega_e \rho_e^{5/2} \tau_e^{3/2}}{\nu' L_{*i}}$

in the first place. Rather, it concentrates on “formation” of large scale structures, from small scales, via quasi-one-dimensional modulations. Using such special forms of modulations makes sense, especially in the context of the dynamics of collapse, or for the later stages of formation, in which the initial isotropy of the seed fluctuations is already broken and there is a selected direction for the modulations. Apparently, the modulations become one dimensional if the initial conditions and the parameters are in an effective “basin of attraction” of the streamer, or zonal flow configurations. Initial direction and localization of the “seed” modulations, as well as the signs of poloidal and radial dispersions are important for the choice of configuration. The stage of formation of streamer, or zonal flow from initially isotropic two-dimensional modulations, will be investigated in detail in a forthcoming article.

The purpose of this work is to elucidate the general mechanism responsible for the formation and stability of streamers and attempt to answer questions such as: “are streamers robust secondary structures?” which requires investigation of the role of nonlinear dynamics, and especially that of collapse. Such an investigation will clarify if there

exists or not, a strong nonlinear tendency toward the formation of the streamer as a singular shear layer. To accomplish this, a simple approximation to the large scale ETG dynamics, which takes nonlinearity into account, is derived and analytically investigated. Understanding the behavior of such simple approximations, given that they do not physically contradict the initial model, is necessary (but not sufficient, of course), for understanding the behavior of the real, complex physical system.

The basic models of ETG and TR turbulence are introduced in Sec. I, together with a very brief review of the perturbation method. In Sec. II, we apply the method to a subcritical primary mode, for which we take the inviscid limit. For this limit we derive the relevant amplitude equation. In Sec. III, and its first three subsections, this equation is solved in various forms. In Sec. III D the Lagrangian structure of the amplitude equation is investigated and conditions for wave collapse are found. The collapse criterion establishes a regime in which a thin, extended streamer cell is a robust attractor for secondary instability. In Sec. IV, we consider a marginally unstable primary mode, with the viscous model equations, and derive an amplitude equation for this limit. In Sec. V we write down a dynamical system model, for this equation, and investigate basic equilibrium and stability of this dynamical system. Section VI contains conclusion and discussions. The Appendix lists the coefficients of the amplitude equations that are derived.

## B. Background

The free energy source for the ETG mode is the electron temperature gradient. The dynamics of the simple plane model of toroidal ETG mode can be described by a set of reduced fluid equations, which include viscosity and thermal diffusivity as simplified models of dissipation,

$$(\partial_t + \hat{z} \times \nabla \Phi \cdot \nabla)(1 - \nabla^2)\Phi + \partial_Y(\Phi + P) + \nu \nabla^4 \Phi = 0, \quad (1)$$

$$(\partial_t + \hat{z} \times \nabla \Phi \cdot \nabla)P - r \partial_Y \Phi - \chi \nabla^2 P = 0. \quad (2)$$

These are the equations of vorticity and pressure with the scalings,

$$r = \frac{\epsilon_B \epsilon_{*e}}{\epsilon_{*i}^2},$$

$$\Phi = \frac{1}{\epsilon_{*i}} \frac{e\phi}{T_i},$$

$$P = \frac{\epsilon_B P}{\epsilon_{*i}^2 P_{i0}},$$

$$t = \epsilon_{*i} \Omega_i t',$$

$$\mathbf{x}' = \frac{\mathbf{x}}{\rho_e \sqrt{\tau_e}},$$

with

$$\epsilon_{*i} = \frac{\rho_e \sqrt{\tau_e}}{L_n},$$

$$\epsilon_B = \frac{\rho_e \sqrt{\tau_e}}{L_B},$$

and

$$\epsilon_{*e} = \frac{\rho_e \sqrt{\tau_e}}{L_p},$$

where

$$L_n = \left( \frac{1}{n_0} \frac{dn_0}{dx} \right)^{-1},$$

$$L_B = \left( \frac{1}{B} \left( \hat{\mathbf{x}} \frac{d}{dx} \right) \times (\mathbf{B}) \cdot \hat{\mathbf{y}} \right)^{-1},$$

and

$$L_P = \left( \frac{1}{P_{0i}} \frac{dP_{0e}}{dx} \right)^{-1}$$

are the background gradient scale lengths. Viscosity, and thermal diffusivity are also nondimensionalized in accord with the space–time scaling that is used, i.e.,

$$\nu = \frac{\nu'}{\tau_e \rho_e^2 \Omega_e \epsilon_{*i}}$$

and

$$\chi = \frac{\chi'}{\tau_e \rho_e^2 \Omega_e \epsilon_{*i}}.$$

Here the primed coordinates are the actual coordinates that has their corresponding dimensions, and  $\rho_e$  is the electron Larmor radius and  $\tau_e = T_i/T_e$  is the ratio of ion to electron temperatures. This is the scaling that is used, mainly for the inviscid limit, which is obtained by taking the limit  $\{\chi, \nu\} \rightarrow 0$ .

Similarly, the free energy source for the TR mode is the temperature gradient acting via buoyancy forces. It is a rotating convection model, which is usually employed for laboratory experiments in a rapidly rotating cylindrical annulus.<sup>14</sup> These experiments are motivated by the problems of understanding the Jovian Great Red Spot and band structure, as well as other geophysical applications. In analogy with atmospheric Rossby waves, we also include a term similar to change of density due to divergence of the polarization drift. Physically it corresponds to potential vorticity evolution in the quasigeostrophic approximation,

$$(\partial_t - \hat{\mathbf{z}} \times \nabla \Psi \cdot \nabla)(1 - \nabla^2)\Psi + \eta^* \partial_y \Psi - \mathcal{R} \partial_y \Theta + \nabla^4 \Psi = 0,$$

$$\mathcal{P}(\partial_t - \hat{\mathbf{z}} \times \nabla \Psi \cdot \nabla)\Theta - \nabla^2 \Theta + \partial_y \Psi = 0.$$

These are, the equations of vorticity and entropy written in terms of stream function  $\Psi$  and thermal fluctuations  $\Theta$ . Here  $\mathcal{R}$  and  $\mathcal{P}$  are Rayleigh and Prandtl numbers, and  $\eta^*$  is the rotation parameter.

Large scale structure formation in this model will be discussed using mean field theory, in which the mean flow and the fluctuations are separated by taking  $\Phi = \bar{\Phi} + \tilde{\Phi}$  and  $P = \bar{P} + \tilde{P}$ . Here  $\bar{(\cdot)}$  is the corresponding field, averaged over

rapid dynamical scales, and  $\tilde{(\cdot)}$  is the fluctuating deviation from that mean. We assume that the dynamics of both the small scale waves (i.e., ETG/TR fluctuations) and the mean field, are described by the same model equations. This is valid for the TR mode. For the plasma case, on the other hand, this is a nontrivial assumption. For the ETG, it corresponds to assuming an adiabatic ion response for the flow as well as fluctuations. While valid for ETG when  $k_\perp \rho_i \gg 1$ , such an approximation is invalid for the ITG mode where the electron response to the flow is nonadiabatic. This is one of the main differences from the prior work of Champeaux and Diamond<sup>11</sup> on the envelope equation approach to secondary modulations, in which the fluctuation dynamics (Hasegawa–Mima equation) are modelled differently from the mean, zonal flow dynamics (Euler equation, driven by Reynolds stress). In that case the electron response is Boltzmann for the drift waves, and non-Boltzmann for the mean flow. In the simple mean field picture, fluctuations in the form of ETG or TR waves are considered to be advected by the mean flow, and the mean flow is driven by the Reynolds stress generated by the small scale  $\tilde{\mathbf{E}} \times \mathbf{B}$  (geostrophic) motion, i.e.,

$$\partial_t(1 - \nabla^2)\bar{\Phi} + \partial_y(\bar{\Phi} + \bar{P}) + \nu \nabla^4 \bar{\Phi} = \overline{\hat{\mathbf{z}} \times \nabla \tilde{\Phi} \cdot \nabla \nabla^2 \tilde{\Phi}}, \quad (3)$$

$$\partial_t \bar{P} - r \partial_y \bar{\Phi} - \chi \nabla^2 \bar{P} + \overline{\hat{\mathbf{z}} \times \nabla \tilde{\Phi} \cdot \nabla \tilde{P}} = 0. \quad (4)$$

We restrict ourselves to consideration of modulations of quasi monochromatic waves, i.e., waves of the form

$$\tilde{\Phi} = \Phi(\bar{X}, \bar{T}, \tau) e^{i(\mathbf{k}\bar{\mathbf{x}} - \omega\bar{t})} + \text{c.c.}, \quad (5)$$

where the complex amplitude changes on scales slower than the fluctuations, for which the general linear dispersion relation,

$$\omega^2(1 + k^2) - \omega k_y + r k_y^2 - \chi \nu k^6$$

$$+ i k^2(\chi(\omega(1 + k^2) - k_y) + \nu k^2 \omega) = 0$$

has two roots

$$\omega_{\pm} = \frac{k_y}{2(1 + k^2)} (1 \pm \sqrt{1 - 4r(1 + k^2)})$$

in the inviscid limit. For the inviscid limit, linear stability is determined by the interplay between different types of advection, represented by the value of the parameter  $r$  for each wave number. For this limit we assume the linear mode is either stable or saturated or that the linear growth is slower than the nonlinear growth. This restriction is imposed in order to isolate nonlinear dynamics. Notice that as  $\epsilon_{*e} \epsilon_B \rightarrow 0$  (i.e., no ETG limit),  $\omega_+$  turns into the inviscid drift wave frequency, and  $\omega_-$  vanishes. On the marginal stability curve  $\omega_+ = \omega_-$ , and above the marginal stability curve the  $\omega_-$  branch decays, while the  $\omega_+$  branch grows.

The back reaction of the flow on the fluctuations may be treated as a perturbation to the linear dispersion relation. The nonlinear problem with coupled fields may be reduced to a single equation for a single field by neglecting nonlinearities involving second and higher order derivatives of the mean fields,

$$\mathcal{L}\tilde{\Phi} + \mathcal{N}(\bar{\Phi}, \bar{P})\tilde{\Phi} = 0. \quad (6)$$

To construct a quasilinear theory taking the dominant nonlinear effects into account, we need to choose an expansion of the fields such that the slow dynamics of the fluctuation amplitude is of the same order (in order parameter  $\epsilon$ ) with these nonlinearities. This is a well known approximation method<sup>15</sup> in fluid dynamics. In practice, the expansion of the dynamical variables,

$$\begin{aligned} \mathbf{x} &= \tilde{\mathbf{x}} + \epsilon^{-1} \bar{\mathbf{X}} \Rightarrow \nabla = \tilde{\nabla} + \epsilon \bar{\nabla}, \\ t &= \tilde{t} + \epsilon^{-1} \bar{T} + \epsilon^{-2} \tau \Rightarrow \partial_t = \tilde{\partial}_t + \epsilon \bar{\partial}_T + \epsilon^2 \partial_\tau, \end{aligned} \quad (7)$$

can be regarded as a Taylor series expansion of the operator  $\mathcal{L}$ , around the point  $(\tilde{\nabla}, \tilde{\partial}_t)$  to a neighboring point  $(\tilde{\nabla} + \epsilon \bar{\nabla}, \tilde{\partial}_t + \epsilon \bar{\partial}_T + \epsilon^2 \partial_\tau)$  in the operator space. The zeroth order term, in this expansion,  $\mathcal{L}_0 = \mathcal{L}(i\mathbf{k}, -i\omega) = 0$ , corresponds to the linear dispersion relation, and the first order term,

$$\begin{aligned} \mathcal{L}_1 &= \left( \frac{\partial \mathcal{L}_0(\tilde{\nabla}, \tilde{\partial}_t)}{\partial(\tilde{\partial}_t)} \bar{\partial}_T + \frac{\partial \mathcal{L}_0(\tilde{\nabla}, \tilde{\partial}_t)}{\partial(\tilde{\nabla})} \cdot \bar{\nabla} \right) \\ &= \frac{\partial \mathcal{L}_0(i\mathbf{k}, -i\omega)}{\partial(-i\omega)} (\bar{\partial}_T + \mathbf{v}_g \cdot \bar{\nabla}) \end{aligned} \quad (8)$$

corresponds to group motion and can be eliminated by transforming to the group velocity frame. As a result, the perturbative feedback of the mean flow on the fluctuations, can be written as a deformation of the wave packet as it travels with its group velocity, i.e.,

$$\mathcal{L}_2 \Phi + \mathcal{N}(\bar{\Phi}, \bar{P}) \Phi = 0. \quad (9)$$

This can be written as

$$\{i(\partial_\omega \mathcal{L}_0) \bar{\partial}_T + (1/2)(\partial_\omega \mathcal{L}_0)(\partial_{\mathbf{k}} \cdot \nabla)^2 \omega(k) + \mathcal{N}(\bar{\Phi}, \bar{P})\} \Phi = 0. \quad (10)$$

## II. INVISCID LIMIT

The inviscid problem may be written in a compact form as in (6)

$$\mathcal{L} = \partial_t [\partial_t (1 - \nabla^2) + \partial_y] + r \partial_{yy},$$

$$\mathcal{N}(\bar{\Phi}, \bar{P}) = \nabla \cdot [\hat{\mathbf{z}} \times \bar{\nabla} \bar{\Phi} (\partial_t (1 - 2\nabla^2) + \partial_y) + \hat{\mathbf{z}} \times \bar{\nabla} \bar{P} \partial_y],$$

with an expansion of the dynamical variables as in (7). We will denote the slow modulation simply by  $\nabla$  from this point on, since  $\bar{\nabla}$  becomes  $i\mathbf{k}$ , when it acts on the quasiharmonic wave (5). The first term, in the expansion of the linear operator  $\mathcal{L}$ , is the dispersion relation  $\mathcal{L}_0 \bar{\Phi} = -[\omega^2 (1 + k^2) - \omega k_y + r k_y^2] \bar{\Phi} = 0$ . We can pass to the group velocity frame ( $\partial_T \rightarrow -\mathbf{v}_g \cdot \nabla$ ) to eliminate  $\mathcal{L}_1$  as described before. Then, the modulation of the wave packet in a frame of reference moving with the group velocity, is described by

$$\begin{aligned} &\{i(\partial_\omega \mathcal{L}_0) \bar{\partial}_T + (1/2)(\partial_\omega \mathcal{L}_0)(\partial_{\mathbf{k}} \cdot \nabla)^2 \omega \\ &- [(\omega(1 + 2k^2) - k_y) \bar{\nabla} \bar{\Phi} - k_y \bar{\nabla} \bar{P}] \times \hat{\mathbf{z}} \cdot \mathbf{k}\} \Phi = 0. \end{aligned} \quad (11)$$

For the inviscid limit we pick a monochromatic wave, for which  $\bar{P}_k = -(r k_y / \omega) \bar{\Phi}_k$ , as the zeroth order primary wave solution. This choice corresponds either to a linearly stable mode, in which case it needs to be driven by other modes, or

more likely to a saturated most unstable mode at mixing length levels for which the nonlinear growth has become large compared to the linear growth. Then we expand the fields as,

$$\tilde{\Phi} = \bar{\Phi}_0 + \epsilon \tilde{\Phi}_1 + \dots, \quad (12)$$

$$\bar{\Phi} = (\epsilon \bar{\Phi}_1 + \epsilon^2 \bar{\Phi}_2 + \dots). \quad (13)$$

Notice that  $\tilde{\Phi}_0 \sim O(1)$  implies  $e\Phi/T_i \sim O(\epsilon_{*i}) \sim O(\sqrt{\tau_e} \rho_e / L_n)$ , which correspond to a *finite but small amplitude ordering for the electrostatic potential*. Indeed, this ordering sets the primary fluctuation level of order the mixing length prediction. However, in this expansion, as  $\epsilon \rightarrow 0$  the solution does not vanish. Instead, the exact, unmodulated travelling wave solution,  $\Phi = (\Phi_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} + \text{c.c.})$ , with  $P_0 = -(r k_y / \omega) \Phi_0$  is recovered. This is possible because both Reynolds stress nonlinearities in the model vanish for a travelling wave solution of this form. The expansion that we employ perturbs this exact solution by introducing a slow spatial and temporal dependence on its amplitude, and separating  $P$  and  $\Phi$ . Then the correction to the vorticity and thermal transport due to these modulations are Reynolds stress and the advection of pressure nonlinearities,

$$\overline{\hat{\mathbf{z}} \times \nabla \tilde{\Phi} \cdot \nabla \nabla^2 \tilde{\Phi}} = 2(\mathbf{k} \cdot \nabla)(\hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla) |\Phi|^2, \quad (14)$$

$$\overline{\hat{\mathbf{z}} \times \nabla \tilde{P} \cdot \nabla \tilde{\Phi}} = -\frac{r k_y}{\omega^2} (\hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla) \left( \partial_T + \frac{\omega}{k_y} \partial_Y \right) |\Phi|^2, \quad (15)$$

where,  $\Phi$  on the right-hand side is the complex amplitude of the fluctuation part defined in (5). It is usually accepted that the term (14), results in large scale structure formation via inverse cascade. Recent investigations<sup>10,16</sup> of the term (15), on the other hand, point to the possibility of large scale structure formation via this term. This may be relevant to our study (especially for the viscous case) because it is this second term, which arises from the phase difference between the two fields, that is the dominant nonlinear drive for the viscous case. These expressions for the Reynolds stresses can be used to write the mean field equations,

$$(\epsilon \partial_T - \mathbf{v}_g \cdot \nabla) \bar{\Phi} + \partial_Y (\bar{\Phi} + \bar{P}) = 2(\hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla)(\mathbf{k} \cdot \nabla) |\Phi|^2,$$

$$\begin{aligned} (\epsilon \partial_T - \mathbf{v}_g \cdot \nabla) \bar{P} - r \partial_Y \bar{\Phi} &= \frac{r k_y}{\omega^2} (\hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla) \left( \mathbf{v}_g - \frac{\omega}{k_y} \hat{\mathbf{y}} \right) \\ &\cdot \nabla |\Phi|^2. \end{aligned}$$

Taking the streamer limit (i.e.,  $\partial_X \rightarrow 0$ ) simplifies these equations considerably. It should be noted here that the time dependence of the mean flow can be neglected for a modulational stability calculation, in which the mean flow is slaved to small scale dynamics. However, it cannot be neglected when the streamer growth is absolute (i.e.,  $\gamma_{\text{str}} \geq \mathbf{v}_g \cdot \mathbf{q}$ ). The result is similar to the subsonic limit of the Zakharov model of Langmuir turbulence. Thus dropping time dependence we can solve for  $\bar{\Phi}$  and  $\bar{P}$ , and obtain

$$\bar{\Phi} = -\frac{r(k_y v_{gy} - \omega) + 2k_y v_{gy} \omega^2}{(v_{gy} - v_{\phi+})(v_{gy} - v_{\phi-}) \omega^2} k_x \partial_Y |\Phi|^2, \quad (16)$$



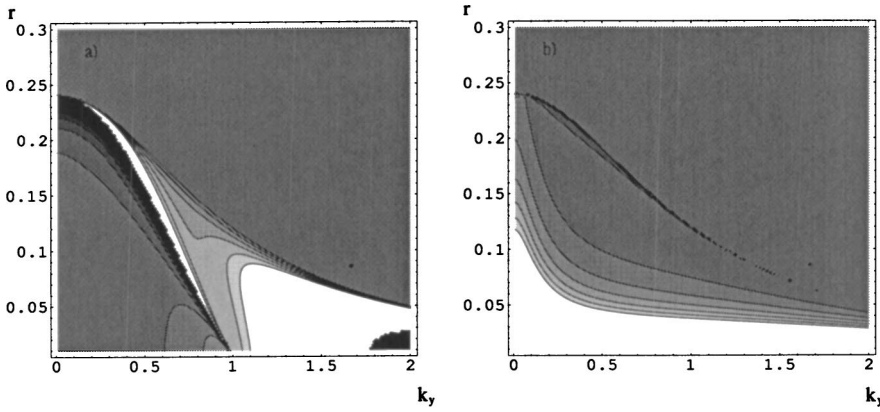


FIG. 1. Contour plots of the dimensionless function  $\delta = -2\alpha/\beta$  as a function of  $r$  vs  $k_y$  with  $k_x = 0.2$  fixed. Figures on the left and right correspond to  $\omega_+$  and  $\omega_-$  branches, respectively. White regions indicate larger values of  $\delta$ , and instability to infinitesimal perturbations.

$$\bar{P} = \frac{2rk_y\omega^2 + (1 - v_{gy})r(k_y v_{gy} - \omega)}{(v_{gy} - v_{\phi+})(v_{gy} - v_{\phi-})\omega^2} k_x \partial_Y |\Phi|^2. \quad (17)$$

Here  $v_{\phi\pm}$  are the phase speeds in the long wavelength limit,

$$v_{\phi\pm} = \frac{1}{2}(1 \pm \sqrt{1 - 4r}).$$

Substituting (16) and (17) into (11) we get the evolution equation for the complex wave amplitude,

$$i\partial_\tau \Phi + \beta \partial_{YY} \Phi + \alpha \partial_{YY} (|\Phi|^2) \Phi = 0. \quad (18)$$

This is a derivative nonlinear Schrödinger equation, with a cubic second derivative nonlinearity. Here  $\beta$  is

$$\beta \equiv \frac{1}{2} \frac{\partial^2 \omega}{\partial k_y^2}.$$

This equation is second order in  $\epsilon$ . Notice that the nonlinearity may be either attractive or repulsive depending on the sign of  $\alpha/\beta$  (see Appendix for  $\alpha$ ). Figures 1 and 2 depict

$\delta = -2\alpha/\beta$  as a function of physical parameters  $r$ ,  $k_x$ , and  $k_y$ . White regions in these graphs correspond to the case when the nonlinearity is attractive. In particular, it is possible to recognize the blade-like region, which starts around  $k_y \sim 1$  in the  $\omega_+$  branch, in both figures. Physically, the attractive case corresponds to enhancement of the amplitude via backreaction of mean flow, since the generated mean flow is in a direction that advects the fluctuation dynamics in such a way that it enhances them in the region where the fluctuation amplitude is larger. It is similar to the instability structure leading to the formation of the modon from similar modulations in two dimensions. In the repulsive case on the other hand, the nonlinear effect is reversed, the generated flow is in a direction which advects the fluctuations in a way to disperse them further. The nonlinear term in (18), which involves the curvature of the intensity field is responsible for the attraction. Thus, the collapse mechanism can be identified as an attraction towards the region of largest intensity

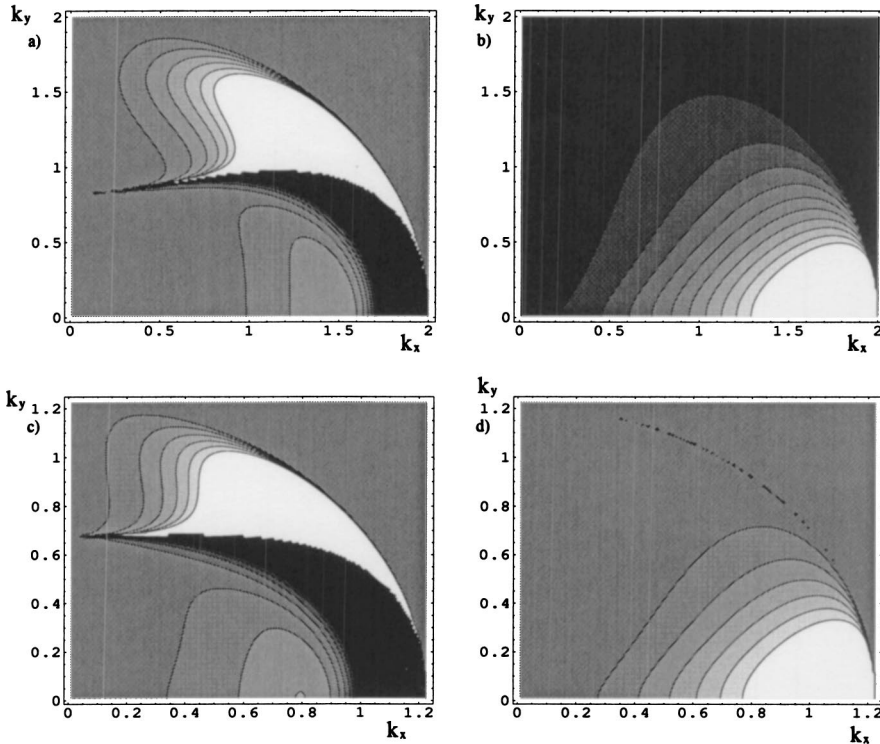


FIG. 2. Contour plots of the dimensionless function  $\delta = -2\alpha/\beta$  as a function of  $k_y$  vs  $k_x$ . (a)  $\omega_+$  mode with  $r = 0.05$ , (b)  $\omega_-$  mode with  $r = 0.05$ , (c)  $\omega_+$  mode with  $r = 0.1$ , (d)  $\omega_-$  mode with  $r = 0.1$ .

curvature (i.e., the turning point), being unbalanced by the linear dispersion. This causes more intensity field, to accumulate near the turning point making it sharper and more effective, which draws more of the intensity field eventually leading to collapse.

At this point it should be pointed out that having two roots in the dispersion relation, and possibility of resonant interaction [enhancement of  $\alpha$  due to resonance  $v_{gy} = v_{\phi\pm} + \delta v(\epsilon)$ ] complicates the issue. In each of these cases, however, there is at least one particular ordering that leads to the same amplitude equation. Physically, this corresponds to the ordering at which the effect of linear dispersion (for either  $\omega_+$  or  $\omega_-$ ) is comparable to the dominant nonlinearities.

### III. PHYSICAL SOLUTIONS

#### A. Linear solutions

Unlike the cubic NLS, (18) has exact linear solutions,

$$\Phi = A_0, \quad (19)$$

$$\Phi = A_0 e^{i(KY - \Omega\tau)}, \quad (20)$$

where the latter one represents the envelope travelling as a wave, with the dispersion relation  $\Omega = \beta K^2$ , which is in fact the second term in the Taylor series expansion of  $\omega(k)$ . Linear stability of these solutions to small amplitude perturbations is a measure of stability of the initial linear quasi harmonic mode. This is mainly determined by the amplitude. For instability,

$$2\alpha A_0^2 < -\beta.$$

Defining  $m \equiv -2\alpha A_0^2/\beta$ , which measures the relative importance of nonlinearity compared to linear dispersion, it is  $m > 1$ . Figures 1 and 2 are contour plots of  $\delta = -2\alpha/\beta$  as  $k_x$ ,  $k_y$ , and  $r$  are varied, hence these contours also depict the stability boundaries as the amplitude  $A_0$  is varied.

#### B. Nonlinear solutions

Besides finite amplitude linear solutions, there are nonlinear solutions of the amplitude equation (18). These solutions may give insight into what sorts of one dimensional structures are possible in ETG turbulence. To find those, we take  $\Phi$  to be of the form,

$$\Phi = A(Y - v\tau) e^{i\phi(Y - v\tau) - i\Omega\tau}$$

and get two coupled ordinary differential equations corresponding to real and imaginary parts of the initial partial differential equation (18). One of these can be solved for  $\phi'$  as a function of the amplitude,

$$\phi'(\xi) = \frac{v}{2\beta} - \frac{C}{A(\xi)^2}. \quad (21)$$

Here ( $\xi \equiv Y - v\tau$ ) and  $C$  is an arbitrary integration constant. We shall set  $C = 0$ , which corresponds to uniform frequency shift. This problem can also be tackled for arbitrary  $C$ , however the results are complicated and do not provide further insight. The real part of (18), may be integrated, with the use of (21), yielding

$$A'(\xi)^2 \left( 1 + 2 \frac{\alpha}{\beta} A(\xi)^2 \right) + NA(\xi)^2 + C^2 A(\xi)^{-2} = H, \quad (22)$$

where  $N = v^2/4\beta^2 + \Omega/\beta$ , and  $H$  is another integration constant related to the energy. By plotting contours of the function  $H(A, A')$ , one may observe the behavior of solutions in the phase space for various values of energy. Both  $N$  and  $\alpha/\beta$  can be positive or negative. These cases should be considered separately.

#### 1. Case 1: $N > 0$ , $\alpha/\beta < 0$ (attractive)

Coefficients in (22) can be absorbed into  $A$  and  $\xi$ , to give

$$A'^2(1 - A^2) + A^2 = \left| \frac{2\alpha}{\beta N} \right| H = m. \quad (23)$$

Using the transformation  $A = \sqrt{m} \sin \phi$  it is possible to solve (23) implicitly,

$$\xi = \int_0^\phi d\phi' \sqrt{1 - m \sin^2 \phi'} = E(\phi|m), \quad (24)$$

where  $m > 0$ . This is the expression for the elliptic integral of the second kind. Let

$$s_m(\xi) \equiv \sin(E^{-1})(\xi|m)$$

be the inverse of the elliptic integral of the second kind, in the same sense as the Jacobi elliptic function  $\text{sn}(\xi|m)$  is the inverse of the elliptic integral of the first kind. Strictly speaking, this is not a well behaved function in certain parts of parameter (i.e.,  $m$ ) range. We introduce it for notational simplicity. It is useful to define the counterparts to this function,

$$c_m(\xi) \equiv \cos \phi(\xi),$$

$$d_m(\xi) \equiv \frac{1}{\sqrt{1 - m \sin^2 \phi(\xi)}}.$$

Using the definitions, various relations between these functions may be obtained,

$$s'_m(\xi) = d_m(\xi) c_m(\xi),$$

$$c'_m(\xi) = -d_m(\xi) s_m(\xi),$$

$$d'_m(\xi) = m d_m^4(\xi) s_m(\xi) c_m(\xi),$$

$$s_m(\xi)^2 + c_m(\xi)^2 = 1,$$

$$d_m(\xi)^2 - m s_m(\xi)^2 d_m(\xi)^2 = 1.$$

The solution  $s_m(\xi)$  [or rather the intensity  $s_m(\xi)^2$ ] can be determined numerically (see Fig. 3) as well. Physically, these solutions represent the one-dimensional limit of a convective cell. For example, the flow profile in Fig. 3(c) looks like a radially elongated version of a series of dipole solutions. As expected, total wave momentum is conserved in these types of flows. It is also noteworthy that (24) is the same expression, which gives the arclength of an ellipse of eccentricity  $m$ , written in terms of  $\phi$ , the complement of the parametric angle.

The solution for the amplitude may be written in the compact form as,

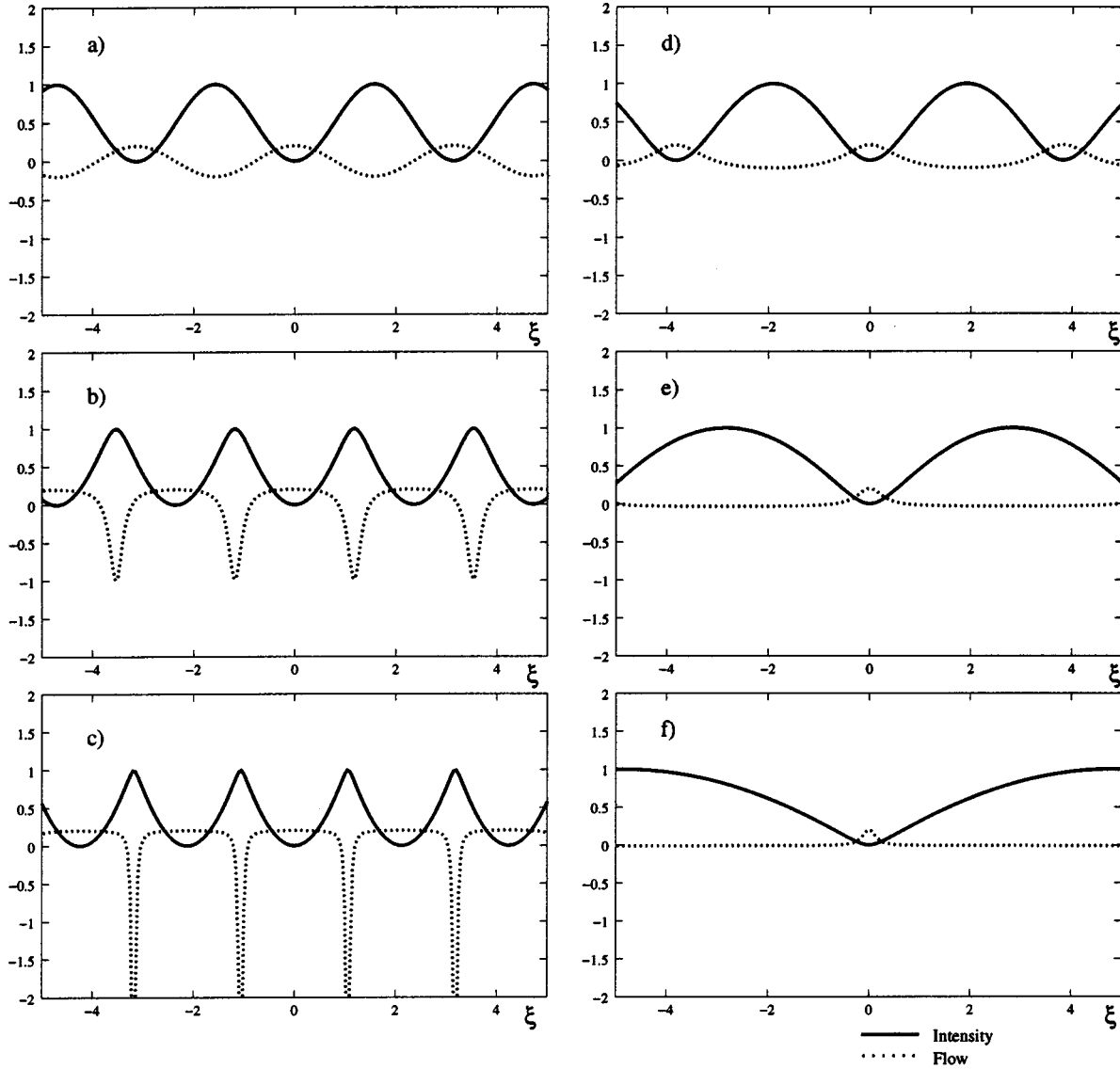


FIG. 3. The solution  $s_m^2(\xi)$  for the envelope intensity (solid line) and the mean flow  $\bar{V} = \hat{z} \times \nabla \bar{\Phi} = -\hat{x} \partial_Y \bar{\Phi}$  (dashed line) for (a)  $m=0.2$ , (b)  $m=0.8$ , (c)  $m \approx 1.0$ , (d)  $m=0$ , (e)  $m=-5$ , and (f)  $m=-20$ .

$$A(\xi) = \sqrt{\frac{H}{N}} s_m(\sqrt{N}(\xi - \xi_0))$$

in terms of unscaled initial variables. The wavelength of which is,

$$\lambda = 4E(m)/\sqrt{N},$$

where  $E(m)$  is the complete elliptic integral of the second kind.

Notice that  $\lambda \rightarrow 2\pi/\sqrt{N}$ , as  $m$  goes to zero. This is same as the “linear” modulated solution, given in (20). Since  $E(m > 0) < \pi/2$ , as the amplitude increases, so does  $m$ , yielding a smaller wavelength.

The shape in Fig. 3(c) corresponds to the indefinitely thin ellipse. It could also be derived by setting  $\partial_{YY}\Phi = 0$  in (18) as,

$$\Phi = \pm \sqrt{\frac{1}{2\alpha} \left( \Omega + \frac{v^2}{4\beta} \right)} (Y - v\tau) e^{i((v/2\beta)Y - (\Omega + (v^2/2\beta))\tau)},$$

This is a linearly increasing or decreasing solution which must turn around in a “mixing layer” of sorts that occurs when the amplitude is large enough (i.e., at  $A^2 = -\beta/2\alpha$ , corresponding to  $m=1$ ). The point at which the function is not smooth corresponds to a small scale turbulent flow region at which the perturbation expansion breaks down.

**2. Case 2:  $N > 0, \alpha/\beta > 0$  (repulsive)**

Equation (23) takes the form

$$A'^2(1 + A^2) + A^2 = m, \tag{25}$$

which has periodic solutions, as can be observed from Fig. 4(a). The solution is written in compact form as

$$A(\xi) = \sqrt{\frac{H}{N}} s_{-m}(\sqrt{N}(\xi - \xi_0)),$$

where

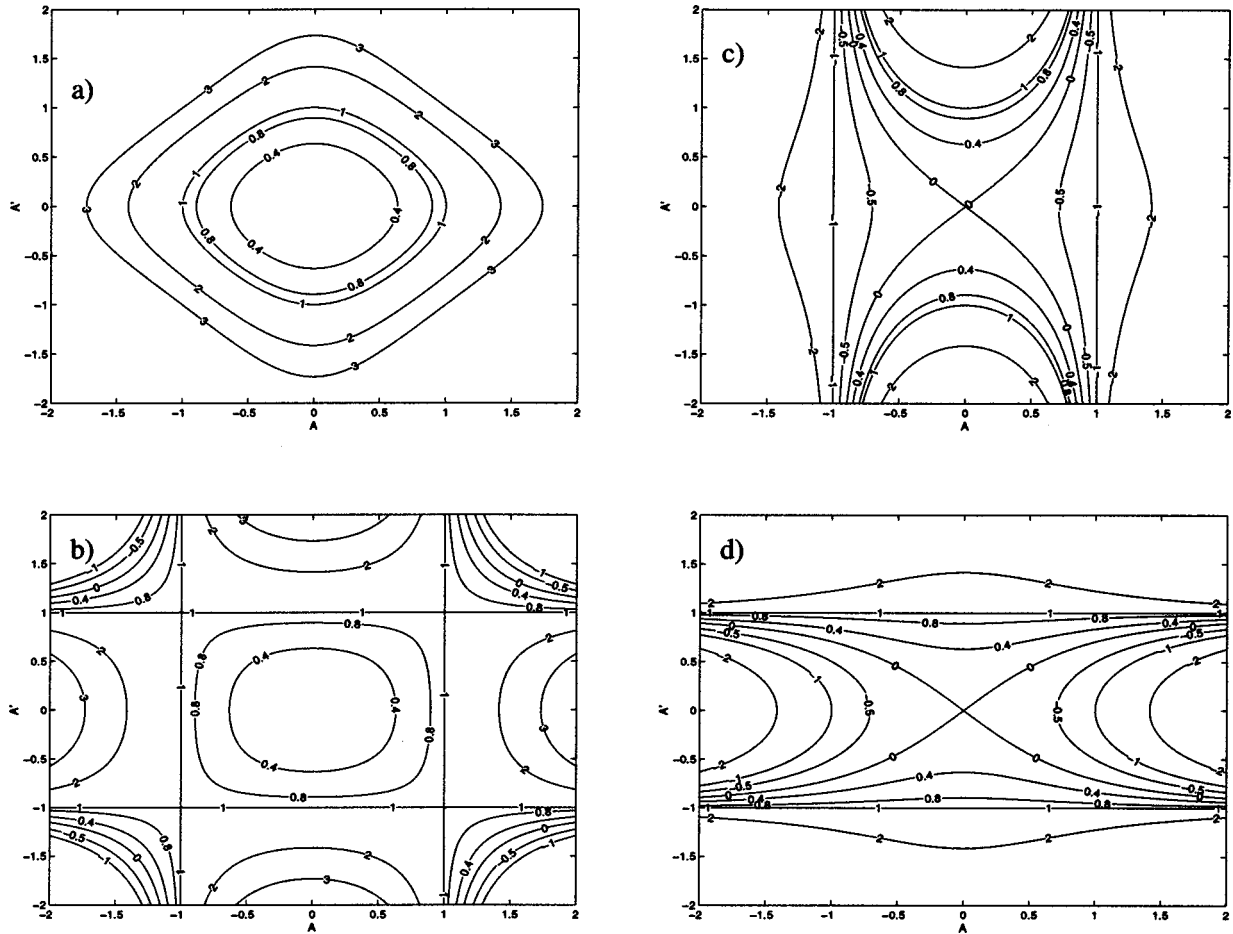


FIG. 4. The phase space plots for the cases (a)  $N > 0$  and  $\alpha/\beta > 0$ , (b)  $N > 0$  and  $\alpha/\beta < 0$ , (c)  $N < 0$  and  $\alpha/\beta < 0$ , and (d)  $N < 0$  and  $\alpha/\beta > 0$ .

$$s_{-m}(\xi) \equiv \sin(E^{(-1)}(\xi | -m)).$$

**3. Case 3:  $N > 0, \alpha/\beta < 0$  (attractive)**

We can write the scaled equation for this case as

$$A'^2(1 - A^2) - A^2 = m.$$

The phase portrait of this equation [i.e., Fig. 4(c)] displays a separatrix at  $m = 0$ . To understand the behavior of the solution on this separatrix, we let  $m = 0$  and solve the resulting equation implicitly,

$$A^2 = \text{sech}^2(\sqrt{1 - A^2} \pm \xi) \tag{26}$$

which can be verified by direct substitution. One can exploit the iterative structure of (26) to visualize intensity (see Fig. 5). Notice that this solution is localized (i.e.,  $A \rightarrow 0$ , as  $\xi \rightarrow 0$ ), and we choose the combination of signs so that the derivative diverges at the origin [correctly so, since  $A'^2(1 - A^2) = A^2$ ,  $A' \rightarrow \infty$  as  $A \rightarrow 1$ , which occurs at the origin]. This localized “thorn” solution may be thought of as a soliton of sorts. The stability properties of this solution should be analyzed further. Notice that our perturbation expansion breaks down around the origin, so if such a solution is to exist, it has to involve dissipative processes (i.e., small scale turbulence) in a mixing layer type region.

The mixing layer idea, is similar to solving the problem with a source located at the origin as a boundary condition.

This solution produces infinite shear at the origin, which is unphysical. In reality, this leads to turbulence near the inflection point. However, this “tendency” might have relevance in the context of shear layer formation.

We ignore the case  $N < 0, \alpha/\beta > 0$  [Fig. 4(d)], which corresponds to repulsive nonlinearity, as it is unphysical.

**C. Mean flow profiles**

Using (16), the mean field for the periodic solutions may be calculated, and is

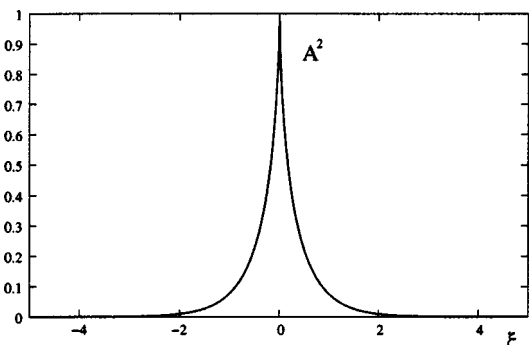


FIG. 5. The localized solution which is not smooth at the origin (i.e., thorn), corresponding to the separatrix of Fig. 4(c).



$$\bar{\Phi} = C(k) \partial_Y |\Phi|^2 = 2C(k) s_m(\xi) c_m(\xi) d_m(\xi),$$

where  $C(k)$  is the constant in (16). The mean flow can then be written as,

$$\bar{V} = 2C(k) d_m^2(\xi) [d_m^2(\xi) c_m^2(\xi) - s_m^2(\xi)] \hat{x}$$

which is already plotted in Fig. 3. Notice that the flow direction depends on the sign of  $C(k)$ .

The mean flow for the thorn solution seems to have vanishing width. In fact the relation between flow and fluctuation amplitudes is not strictly valid around this layer, to a width of order a few  $\rho_e \sqrt{\tau_e}$ , and some dissipative or turbulent behavior should be expected as a result. The simplest way to resolve this layer would be to introduce dissipation and transform to stretched variables to perform a perturbation expansion. This amounts to treating the region around  $x=0$  as a turbulent boundary layer.

#### D. Lagrangian structure of Eq. (18)

The equation describing the evolution of the drift ETG fluctuation envelope has a Lagrangian structure. The Lagrangian density for the case  $\alpha/\beta > 0$  [corresponding to Figs. 4(b) and 4(c)] is

$$\mathcal{L} = \frac{i}{2} (\Phi^* \partial_\tau \Phi - \Phi \partial_\tau \Phi^*) - |\partial_Y \Phi|^2 + \frac{1}{2} (\partial_Y (|\Phi|^2))^2.$$

Nonlinear Schrödinger (NLS) type amplitude equations, with real nonlinearity, conserve total wave intensity,

$$\partial_t |\Phi|^2 + \partial_Y J = 0.$$

Here  $J = i(\Phi \partial_Y \Phi^* - \Phi^* \partial_Y \Phi)$  is the usual Schrödinger current density. The total Hamiltonian,

$$H = \int \mathcal{H} dY = \int [|\partial_Y \Phi|^2 - \frac{1}{2} (\partial_Y (|\Phi|^2))^2] dY \quad (27)$$

and wave momentum are also conserved,

$$\partial_t J + \partial_Y S = 0,$$

where  $S = 4|\partial_Y \Phi|^2 + \partial_{YY} (|\Phi|^2 + |\Phi|^4) - 3(\partial_Y (|\Phi|^2))^2$  is the radiation pressure. The intensity field variance, which is a positive definite quantity related to disturbance width, can be defined as:

$$V = \int |\Phi|^2 Y^2 dY.$$

Using the virial theorem,<sup>17</sup>

$$\frac{d^2 V}{dt^2} = 2 \int S dY = 8H - 2(\partial_Y (|\Phi|^2))^2$$

we conclude that if the nonlinearity is attractive and the initial mean field  $((\partial_Y (|\Phi|^2))^2 \propto \bar{\Phi}^2)$  is strong enough, the width of the wave packet vanishes in finite time. Since the total intensity is conserved, as the width goes to zero, the amplitude must diverge! Thus, within its domain of validity, and given  $\alpha/\beta < 0$  and  $H < 1/4(\partial_Y (|\Phi|^2))^2$ , the model predicts spectral collapse. This suggests a natural tendency to generate radially extended, singular shear layers in ETG/TR

turbulence. Why the initial disturbance can be treated as quasi-one-dimensional is a separate issue and will be discussed in detail in another paper.

The actual picture of the wave collapse in ETG is more complicated. As the collapse starts, a very small shear layer, with a very large mean flow starts to form. Then the shear flow acts on the small scale modes, and modifies them non-perturbatively via shearing feedback. Thus, the collapse of streamer flows is possibly “self-healing.” This self-healing behavior is thus a possible origin of the intermittency in streamer dynamics. Self-healing tendency after spectral collapse indicates cyclic behavior of the streamer at collapse time scales which can be estimated to be around the collapse time of the corresponding NLS [i.e.,  $t_c \approx (1/2)\sqrt{V/H}$ ], however it is also possible that this small scale shearing feedback mechanism does not lead to self-healing, in which case large bursty events of large transport may occur.

#### IV. VISCOUS CASE

The effects of dissipation may be important on relatively small scales, or over a long time. The important caveat to note here for the plasma case is that these scales should be larger than the gyroradius scale for self consistency. The scalings

$$t = \frac{t' v'}{\rho_e^2 \tau_e \Omega_e},$$

$$\Phi = \frac{e \Phi}{T_{i0}} \frac{\rho_e^2 \tau_e \Omega_e}{v'} \leftrightarrow -\Psi,$$

$$P = \frac{1}{\mathcal{P} \epsilon_{*e}} \frac{P}{P_{i0}} \leftrightarrow \Theta,$$

$$\mathcal{P} = \frac{v'}{\chi'},$$

$$\mathcal{R} = \frac{\epsilon_B \epsilon_{*e}}{\chi' v'} \rho_e^4 \tau_e^2 \Omega_e^2$$

and

$$S = \frac{\epsilon_{*i}}{v'} \rho_e^2 \tau_e \Omega_e \leftrightarrow \eta^*,$$

bring these scales into focus. Notice that primed variables are the actual coordinates that have their corresponding dimensions, and the parallel with the TR notation is denoted by double sided arrows. In the TR model such normalization is already implicit, therefore, the scalings that are used for the “viscous” ETG mode, facilitate a one to one correspondance to the “meteorological” notation. The viscous problem in rescaled form,

$$\partial_t (1 - \nabla^2) \Phi + S \partial_y \Phi + \mathcal{R} \partial_y P + \nabla^4 \Phi = \hat{z} \times \nabla \Phi \cdot \nabla \nabla^2 \Phi,$$

$$\mathcal{P} (\partial_t P + \hat{z} \times \nabla \Phi \cdot \nabla P) - \partial_y \Phi - \nabla^2 P = 0$$

may be restated as

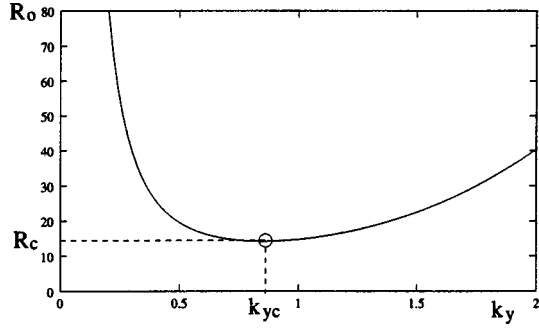


FIG. 6. Marginal stability curve of the primary modes, showing marginal stability parameter vs wave number, around the minimum of which the expansion will be performed.

$$\begin{aligned} \mathcal{L} &= (\mathcal{P}\partial_t - \nabla^2)(\partial_t(1 - \nabla^2) + \mathcal{S}\partial_y + \nabla^4) + \mathcal{R}\partial_{yy}, \\ \mathcal{N}(\bar{\Phi}, \bar{P}) &= -\mathcal{P}[\mathcal{R}k_y \nabla \bar{P} + (\omega(1 + 2k^2) \\ &\quad - \mathcal{S}k_y + ik^4(1 + \mathcal{P}^{-1}))\nabla \bar{\Phi}] \times \hat{\mathbf{z}} \cdot \mathbf{k} \end{aligned}$$

and has the linear dispersion relation,

$$(\mathcal{P}\omega + ik^2)(\omega(1 + k^2) - \mathcal{S}k_y + ik^4) + \mathcal{R}k_y^2 = 0.$$

We can easily identify the most unstable mode for the viscous problem, and make a formal expansion into the unstable region. Here the marginal stability curve is given by

$$\mathcal{R}_0 = \frac{\mathcal{S}^2 k^2}{(k^2 + \mathcal{P}^{-1}(k^2 + 1))^2} + \frac{k^6}{k_y^2}. \quad (28)$$

The marginal stability curve, for a fixed value of  $k_x$  (presumably small), is shown in Fig. 6.

The dynamical variables are expanded as in (7), the fields, however, are expanded as:

$$\tilde{\Phi} = \epsilon^{1/2} \tilde{\Phi}_1 + \epsilon \tilde{\Phi}_2 + \dots,$$

$$\bar{\Phi} = \epsilon \bar{\Phi}_1 + \epsilon^2 \bar{\Phi}_2 + \dots.$$

Since the scaling of the field variable  $\Phi$  is different from the inviscid case, a different expansion is expected. This may be considered as resolving the viscous mixing layer. Comparing the scaling to the inviscid case and assuming that  $e\Phi/T_i$ , the physical field is the same order in both cases, yields

$$\frac{\rho_e^2 \Omega_e \tau_e}{\nu'} \frac{\rho_e \sqrt{\tau_e}}{L_n} \leq \epsilon^{1/2},$$

which makes sense as long as  $\nu$  is not vanishingly small. The average Reynolds drive up to order  $\epsilon^2$  become

$$-(\partial_T + \mathcal{S}\partial_Y)\bar{\Phi} = \mathcal{R}\partial_Y \bar{P},$$

$$\partial_T \bar{P} - \mathcal{P}^{-1} \partial_Y \bar{\Phi} = -\frac{2\mathcal{P}^2 k_y k^2}{\omega^2 + \mathcal{P}^{-2} k^4} \hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla |\Phi|^2.$$

That is, the response of the mean pressure to the ponderomotive drive behaves like a driven wave, when the amplitude is small, but not vanishingly small.

In a manner similar to the inviscid case, the expansion may be represented as a Taylor series expansion of the linear

operator. The central difference is that the stability parameter is also expanded into the region of instability, i.e.,

$$\mathcal{R} = \mathcal{R}_c(1 + \epsilon^2 \zeta), \quad (29)$$

where  $\zeta$  is deviation from marginality,  $\mathcal{R}_c$  is the minimum value of the marginal stability curve (28). Now the Taylor series expansion of the operator, which also depends on the stability parameter, turns into an expansion from the point,  $(\tilde{\nabla}, \tilde{\partial}_t, \mathcal{R}_c)$  to the neighboring point  $(\tilde{\nabla} + \tilde{\nabla}, \tilde{\partial}_t + \tilde{\partial}_T + \partial_\tau, \mathcal{R}_c(1 + \epsilon^2 \zeta))$ . Dropping the bars on the modulational operators, the straightforward procedure yields the evolution equation for the complex amplitude,

$$\begin{aligned} \left\{ i\partial_\tau + \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial k_y^2} - k_y^2 (W + i\Gamma) \frac{d^2 R}{dk_y^2} \right) \partial_{YY} - k_y^2 \mathcal{R}_c \zeta (W + i\Gamma) \right. \\ \left. + (W + i\Gamma)(\beta + i\delta) \partial_Y |\Phi|^2 \right\} \Phi = 0. \quad (30) \end{aligned}$$

This equation is a special case of the generalized complex Ginzburg–Landau equation.<sup>18</sup> The term that is proportional to  $\zeta$  represents linear growth and frequency shift due to the deviation from the marginal stability curve. The constants are given in the Appendix, and  $W$  and  $\Gamma$  are directly proportional to linear growth and frequency shift. Notice that the dispersion term also has a contribution coming from the change of the stability parameter. The last term, depending on the sign of  $\beta$  and  $\delta$ , represents either nonlinear growth or damping. Remarkably, a particular real limit of this equation, (namely,  $W \ll \Gamma$ ,  $R_c \ll 1$  and  $\delta \ll \beta$ ) which also corresponds to the equation for dissipative zonal flows (see Ref. 11) has a form which is invariant under the Talanov “self-focusing” transformation.<sup>19</sup> This implies that if any localized solution to this equation is found, a self-focusing version can also be constructed.

## V. DYNAMICAL SYSTEM ANALYSIS

To analyze the equation,

$$\begin{aligned} i\partial_\tau \Psi + (\alpha + i\beta) \partial_{YY} \Psi + (\omega + i\gamma) \Psi \\ + (\delta + i\eta) \partial_Y (|\Psi|^2) \Psi = 0, \quad (31) \end{aligned}$$

where the Greek letters are generic constants unrelated to the constants used in previous sections, we concentrate on travelling solutions,

$$\Psi(Y, \tau) = A(\xi) e^{-i(\Omega - \omega)\tau + i\phi(\xi)}.$$

Substituting this into (31), and defining

$$q \equiv \phi' \Rightarrow q' = \phi'',$$

$$K = \frac{A'}{A} \Rightarrow K' = \frac{A''}{A} - K^2,$$

we get a dynamical system<sup>18</sup> with a pseudotime variable  $\xi$ ,

$$\begin{aligned} q' &= \frac{1}{\alpha^2 - \beta^2} [\alpha(\gamma - \nu K + 2\eta K A^2) \\ &\quad - \beta(\Omega + \nu q + 2\delta K A^2)] - 2Kq, \end{aligned}$$

$$K' = \frac{-1}{\alpha^2 + \beta^2} [\beta(\gamma - vK + 2\eta KA^2) + \alpha(\Omega + vq + 2\delta KA^2)] - q^2 - K^2,$$

$$A' = KA.$$

Two fixed points of this system with nonvanishing amplitude are  $K=0$ ,  $q = \pm(\gamma/\beta)^{1/2}$ , where  $v = \pm(\alpha\gamma - \beta\Omega)\sqrt{\beta\gamma}$ , with same signs for  $q$  and  $v$ . Notice that the fixed point is independent of the amplitude, therefore the field at the fixed point may have any amplitude. If we concentrate on the nontrivial fixed point (picking all parameters positive as an example) as long as  $\beta v/(\alpha^2 + \beta^2) > 0$  and  $2q_* < \alpha v/(\alpha^2 + \beta^2)$  (where  $q_*$  is the “fixed point,” or the “frequency of the fixed oscillation”) the point is a stable one. In such a case, the amplitude remains constant against infinitesimal background perturbations. The opposite case will correspond to tertiary instability. Since the stability requires the structure to move with a certain fixed velocity,  $v$  may also be interpreted as a dynamical variable of sorts. To determine the stability of a particular physical system, the parameters in (31) should be calculated (see the Appendix), and the analysis above should be repeated for the particular system with corresponding signs.

An approximation to the solution, when the zero amplitude fixed point is unstable and finite amplitude fixed point is stable, is a travelling “kink.” It is possible to imagine a situation in which there are two such kinks of opposite wave number. Since they have opposite wave number, they have opposite velocities as well, thus, they will move towards (or away from) one another, and they will eventually merge (within a system with periodic boundary conditions). Close to the merging point, the dynamical systems method is no longer valid. This is one of the possible scenarios akin to wave collapse in this system.

Also, it is possible to consider the amplitude equation for the viscous case [i.e., (30)] as an equation to be satisfied only in the boundary layer, required by the rapid changes in the derivatives of the solution for the inviscid problem. From this perspective, using linearly stable modes for the inviscid case, but linearly unstable modes for the viscous case can be justified since the linearly unstable modes should be more or less concentrated around the tip of the modulational envelope, and they will ultimately be balanced by the nonlinear and dissipative dynamics that are considered in this section. In general, it should be possible to construct a “global” solution of the problem by matching the solutions in these different layers. In particular, it is possible to find a more general “thorn” solution from (26), by replacing the layer around the inflection point by a constant “fixed point” solution of the viscous problem. In fact, the viscous part of the solution would oscillate around the fixed point since the matching condition is not exact, leading to production of shear. Notice that such a solution is possible only if the velocity of the “thorn” is set by the dissipative dynamics. Modulations that are moving too fast probably disintegrate. The same idea about matching solutions in different layers, can also be applied to the periodic solutions of the inviscid

problem. Thus, from the modulational perspective, the picture of a streamer arises as a compound nonlinear structure, made up of a shear layer and its vicinity.

## VI. DISCUSSION

As a result of the modulational instability method that we used, we obtained relatively simple, approximate analytical equations that describe the evolution of ETG/TR fluctuations, which interact coherently with a mean flow. The general structure of Zakharov equations, which describe the nonlinear evolution of Langmuir waves, and predict wave collapse, also “persists” in ETG/TR case.

The central result of this paper is the derivation of an amplitude equation, which has a Lagrangian structure for the inviscid case. This leads to identification of the large scale secondary instability of the fluctuation amplitude to infinitesimal perturbations. Using this amplitude equation we have constructed a family of nonlinear travelling wave solutions and a localized “thorn” solution. Most importantly, using the Hamiltonian structure of this equation it was possible to identify the conditions for the *wave collapse* in this system (i.e., prove a Talanov theorem). This is an essential result of the paper since it predicts that a sufficiently localized mean flow will trap more and more ETG fluctuations, leading to a finite time singularity. Formation of such singular layers is important for explaining the excess of electron thermal transport. The specific conditions for the collapse, and a basic estimation for the collapse time are also given in the text.

The parameter values, for which the wave collapse occurs, can be identified as the basin of attraction for the generation of a one dimensional small scale flow. Notice that, this basin is not only defined as a range for the physical parameters of the small scale mode ( $r$ ,  $k_x$ , and  $k_y$ ) but also defined as a range for the Hamiltonian, which is a measure of the field energy. This means that the potential well for the ETG fluctuations that result from initial mean flow seeds, should be deep enough so that it can trap fluctuations. Since the mean flow is driven by the fluctuations, this corresponds to having sufficient initial field energy.

The zonal flow, another anisotropic convective cell state with  $k_y \rightarrow 0$ , which is neglected in this analysis may also be present as another attractor, in which case the system may oscillate in a chaotic manner between these two attractor states. It is therefore important to identify which one is the stronger attractor, for the ETG case, and what are the differential effects of physical phenomena that are neglected in this simple picture. This issue will be addressed in a future publication.

Similarly, derivation of the evolution equation for the viscous case leads to a special case of generalized complex Ginzburg–Landau equation. Using the dynamical systems approach we have identified the fixed points of this equation and conditions for their linear stability. The analysis of the viscous case allows the possibility of constructing the streamer as a compound nonlinear structure made up of a dissipative shear layer and its inviscid vicinity. In practice, numerical solutions are possible for both cases and can be compared with direct simulation results to test the model.

Notice that, the form of equations that we have derived, would be relevant for the zonal flows and streamers of Hasegawa–Mima-type drift dynamics (with absolute or modulational growth, as can easily be shown), if a completely adiabatic electron response can be assumed. The form of (18) is a consequence of the complete adiabatic ion response.

The role of the decaying branch in the ETG dynamics has not yet been fully confronted. This can be approached via two nonlinearly coupled amplitude equations using the framework developed here. Even in the weak turbulence limit, the role of this coupling can be important, especially near marginality. Therefore it needs to be investigated thoroughly. Many of the ideas that are developed in this paper can, in principle, be carried over to two dimensions. This is important since a meaningful estimation of the radial scale of the streamer as a nonlinear large scale structure, and of the resulting transport, can only be made when the radial saturation mechanism is thoroughly understood.

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## APPENDIX: COEFFICIENTS OF THE AMPLITUDE EQUATIONS

The parameter  $\alpha$  that is used in (18) can be found by substituting (16) and (17) into (11),

$$\alpha = \frac{k_x^2}{(2\omega(1+k^2) - k_y)} \times \left[ \frac{(\omega(1+2k^2) - k_y v_{gy})(r(\omega - v_{gy} k_y) - 2\omega^2 k_y v_{gy})}{(r - v_{gy}(1 - v_{gy}))\omega^2} - 2k_y^2 \right].$$

The parameters for the viscous case can also be found in a parallel manner, except that in this case,  $\partial_\omega \mathcal{L}_0$  (whose in-

verse correspond to  $W$  and  $\Gamma$ ) and the operator representing the nonlinearity (corresponding to  $\beta$  and  $\delta$ ) are complex. Using the condition of marginality (i.e.  $\mathcal{P}^2 \omega^2 + k^4 = R_0 k_y^2 / k^2$ ) and the dispersion relation for further simplification,

$$W = \frac{\mathcal{P}(Sk_y - 2\mathcal{P}\omega(1+k^2))}{\mathcal{P}^2(2\omega(1+k^2) - Sk_y)^2 + k^4(1+k^2 + \mathcal{P}k^2)^2},$$

$$\Gamma = \frac{k^2(1+k^2 + \mathcal{P}k^2)}{\mathcal{P}^2(2\omega(1+k^2) - Sk_y)^2 + k^4(1+k^2 + \mathcal{P}k^2)^2},$$

$$\beta = \frac{2\mathcal{P}^2 k^4 (\omega(1+2k^2) - v_{gy} k_y) \frac{k_x^2}{k_y^2}}{\mathcal{P}(S - v_{gy}) - R_0 \frac{k_x^2}{k_y^2}},$$

$$\delta = - \frac{2\mathcal{P}^2 k^8 (P^{-1} + 1) \frac{k_x^2}{k_y^2}}{\mathcal{P}(S - v_{gy}) - R_0 \frac{k_x^2}{k_y^2}}.$$

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